Attitude Control of Flexible Space Vehicles

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Objectives of this paper are to enhance the concept of multiple control in attitude control systems for large flexible space vehicles and to discuss some questions of stability and structure-control system interaction. The approach to the design of a multiple control system is illustrated with a typical example for which the control system configuration includes a main loop providing control for the main portion of the structure and auxiliary loops providing active damping for the flexible appendages. Criteria for the first cut design of each auxiliary loop are given. The performance of the multiloop control system described, vs that of the corresponding single-loop, is illustrated by the results of analog simulation. Qualitative results of analysis and stability limitations of different classes of attitude control system configurations for large flexible space vehicles are also discussed. The normal mode approach particularly suitable for the use of transfer functions is considered in modeling the structure dynamics. A criterion for mode separability in a multimode structure is presented. The simplification deriving from its application to the analysis of an attitude control system is illustrated.

Nomenclature

J_i = moment of inertia
$K_{p,q}{}^i$ = admittance for the ith mode
T_p = torque applied at point P
θ_q = rotation at point Q
ω_i = undamped natural frequency for the ith mode
ζ_i = damping ratio for the ith mode
ω_i' = undamped natural frequency of the associated zero
ζ_i' = damping ratio of the associated zero
α_i = damping coefficient
k_i = spring constant
s = Laplace operator
Σ = summation operator
Π = multiplication operator

Introduction

INCREASING attention has been focused, in recent years, on the problem of designing attitude control systems for large, flexible space vehicles. This problem presents two basic subproblems. One is concerned with providing a mathematical model for the dynamic analysis of a flexible structure, a problem which has received greater attention and produced a considerable number of authoritative papers and reports.[1-10] The other concerns the design of the over-all attitude control system for highly flexible spacecraft so that the required system performance is met without the surge of dangerous interactions with the structure dynamics, objectives which are sometimes conflicting for a given control system configuration.

The problem of interaction between attitude control systems and flexible structures, whose undesirable effects can vary from large attitude errors to large structural deflections or total control system instability, has also been the object of investigation in recent papers and reports.[1,4,11-13]

As the spacecraft structures have become larger and more flexible, the traditional attitude control system design approach, based on a rigid body approximation of the structure dynamics, has been recognized inadequate to ensure satisfactory results. Attention has, therefore, been concentrated on the coupling of the vibrational dynamics of the structure into the dynamics of the attitude control system.

This paper begins with a discussion of some analytical aspects of the problem of interactions between an attitude control system and a flexible spacecraft. The modal approach, particularly useful for its amenability to a transfer function representation of the structure when small oscillations are assumed, is used throughout the paper. For a multimode structure, a criterion of mode separability, proved in the Appendix, is shown to be very effective in simplifying the analysis. It is shown, in fact, that when mode separability applies, the effect of each mode on the stability of the attitude control system can be evaluated separately. The arguments discussed are, therefore, illustrated on the basis of a typical control system configuration having a structure with one highly underdamped oscillatory mode whose frequency is scanned over and above the control system bandwidth.

It is discussed qualitatively, in the light of the Nyquist analysis techniques, how the system stability is affected when the sensors and actuators are located on the same rigid section of the structure or when they are located on parts of the structure separated by flexible connection.

For a single loop attitude control system, it is shown that, although stability may not be affected when some of the modal frequencies are well within the control system bandwidth, dangerous sustained oscillations can be excited in these modes by interaction with the control system, unless the structural damping is increased either by passive means or by active auxiliary control loops. Attention is, therefore, given to the second alternative and rule-of-thumb design criteria are prospected. The arguments are illustrated with the analog simulation results of a typical example.

System Dynamics and Stability

It is chosen to represent the structure dynamics by the normal mode model, as shown in Fig. 1, which illustrates the transfer function relating forces or torques applied at a point P of the vehicle to displacements or rotations at a point Q (Ref. 1, Sec. V.B). From basic theory on the dynamics of structures it can be shown that the "admittances" $K_{p,q}{}^i (i = 1 \ldots n)$ of Fig. 1 are all positive if P and Q coincide or belong to the same rigid component of the structure and single axis motion of only one kind is considered, such as rotational motion. In this case, the criterion that follows, proved in the Appendix, provides a considerable simplification in the analysis and design of a related attitude control system.

Criterion

When $K_{p,q}{}^i/K_{p,q}{}^0 \ll 1$, $i = 1, 2, \ldots n$, the effect of each mode

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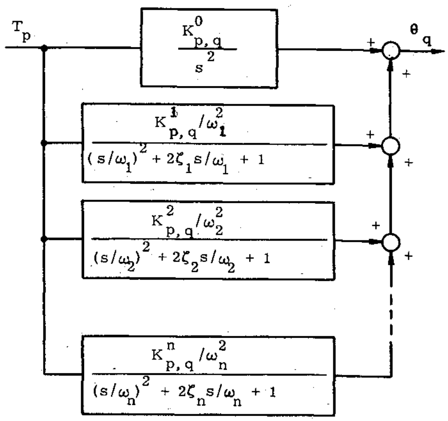

Fig. 1 Modal representation of structure dynamics.

on the stability of the control system can be evaluated separately in first approximation. Thus, for simplicity and exemplification, the qualitative analysis that follows is based on the single-mode structural model of Fig. 2, representing two masses connected by a lossy torsional spring and free to oscillate about the common axis of symmetry. If the torques T_0 and T_1 are applied to the masses of moments of inertia J_0 and J_1, the equations of motion are

$$\begin{aligned} J_0\ddot{\theta}_0+\alpha_1(\dot{\theta}_0-\dot{\theta}_1)+k_1(\theta_0-\theta_1) &= T_0 \\ J_1\ddot{\theta}_1+\alpha_1(\dot{\theta}_1-\dot{\theta}_0)+k_1(\theta_1-\theta_0) &= T_1 \end{aligned} \tag{1}$$

where k_1 is the spring constant and α_1 is the equivalent

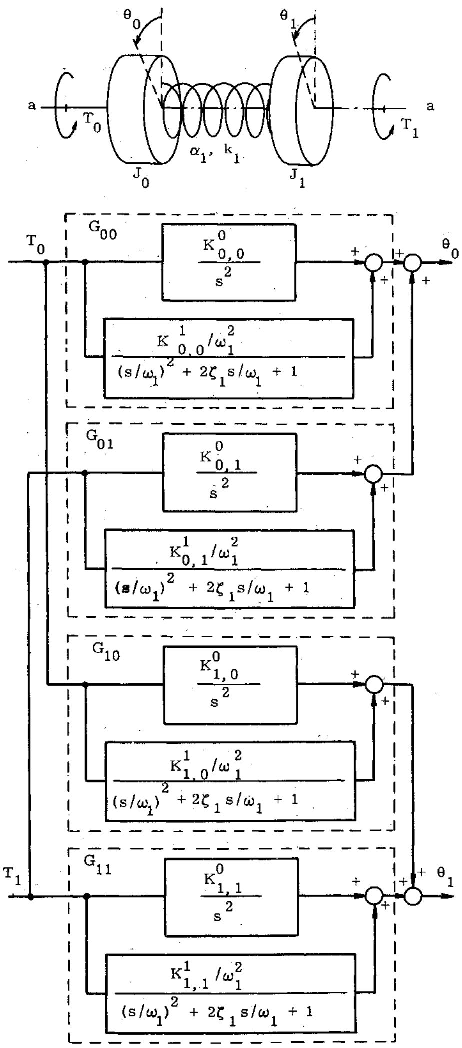

Fig. 2 Single-mode model.

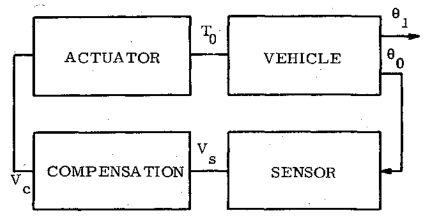

Fig. 3 Control system configuration.

damping coefficient. Solving (1) for θ_0 and θ_1 yields the transfer function matrix

$$\begin{bmatrix}\theta_0\\ \theta_1\end{bmatrix}=\begin{bmatrix}G_{00} & G_{01}\\ G_{10} & G_{11}\end{bmatrix}\begin{bmatrix}T_0\\ T_1\end{bmatrix} \tag{2}$$

where $G_{i,j}(i,j=0,1)$ are the transfer functions shown in Fig. 2 and the various parameters appearing in them have the following expressions:

$$\begin{aligned} &\omega_1 = [k_1(J_0+J_1)/J_0J_1]^{1/2}; \quad \zeta_1=\alpha_1\omega_1/2k_1 \\ &K_{0,0}{}^0=K_{0,1}{}^0=K_{1,0}{}^0=K_{1,1}{}^0=1/(J_0+J_1) \\ &K_{0,0}{}^1=(J_1/J_0)/(J_0+J_1); \quad K_{1,1}{}^1=(J_0/J_1)/(J_0+J_1) \\ &K_{0,1}{}^1=K_{1,0}{}^1=-1/(J_0+J_1) \end{aligned} \tag{3}$$

It is assumed that the attitude of the vehicle schematically represented in Fig. 2 is to be controlled about the axis $a-a$ by the control system shown in Fig. 3 in which no active control is exerted on body J_1. Since $T_1=0$, only the transfer function G_{00} is relevant in the stability analysis. Thus, the above-mentioned criterion applies when $J_1 \ll J_0$. Neglecting, in the first cut design, the modal component in the vehicle transfer function θ_0/T_0 a satisfactory control loop operation is readily obtained with a lead compensation network. The resulting Nyquist diagram of the open loop transfer function is shown qualitatively in Fig. 4 which also shows the high-frequency effects of time lags in the sensor and actuator. The effects of the modal frequency are best evaluated by expressing the transfer function θ_0/T_0 in the form

$$\frac{\theta_0}{T_0}=\frac{K_{0,0}{}^0}{s^2}\left(\frac{s^2/\omega_1'^2+2\zeta_1's/\omega_1'+1}{s^2/\omega_1{}^2+2\zeta_1 s/\omega_1+1}\right) \tag{4}$$

$$\omega_1'=\omega_1[K_{0,0}{}^0/(K_{0,0}{}^0+K_{0,0}{}^1)]^{1/2}; \quad \zeta_1'/\omega_1'=\zeta_1/\omega_1 \tag{5}$$

which is obtained by adding the rigid body and the modal component of the transfer function G_{00}. Since $\omega_1'<\omega_1$, the modal contribution (factor in parentheses) to the open-loop transfer function is an antiresonance peak followed by a resonance peak at slightly higher frequency. The effect on the

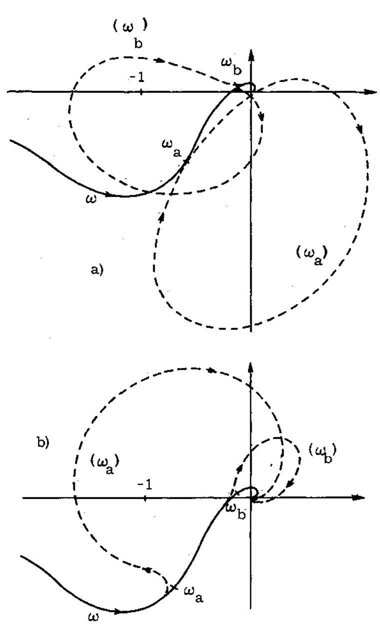

Fig. 4 Nyquist diagrams.

Nyquist diagram is a magnitude decrease, while the phase advances rapidly with a maximum swing close to $+180°$, followed by magnitude increase and phase swinging back to a net zero asymptotic contribution. This is illustrated in Fig. 4a for two different modal frequencies ω_a and ω_b. It is clear from the figure that control system instability (encirclement of the point $-1+j0$) can only occur if the system is conditionally stable, as the typical case of Fig. 4a shows, and the modal frequency ranges around the intersection of the rigid mode Nyquist diagram with the negative real axis (case ω_b). In such case, however, stability can be preserved by additional compensation providing suitable increase of the gain margin. The occurrence of a modal frequency in the lower range of frequencies does not produce an unstable condition. But stability is not the only question. A modal frequency located within the control system bandwidth can easily be excited during transient operation. The amplitude of the oscillations induced in the appendage (mass J_1 in the present example) depends on the open loop transfer function θ_1/T_0 of the appendage and on the frequency content of T_0. From Fig. 2 and expressions (5)

$$\frac{\theta_1}{T_0} = \frac{K_{1,0}^{\,0}}{s^2}\left(\frac{2\zeta_1' s/\omega_1'+1}{s^2/\omega_1^{\,2}+2\zeta_1 s/\omega_1+1}\right) = \frac{K_{0,0}^{\,0}}{s^2}\left(\frac{2\zeta_1 s/\omega_1+1}{s^2/\omega_1^{\,2}+2\zeta_1 s/\omega_1+1}\right) \qquad (6)$$

For very low damping ($\zeta_1 \ll 1$), the corner frequency $\omega_1/2\zeta_1$ in the numerator of (6) is much greater than ω_1 and its effect on the frequency response function can therefore be neglected. If T_0 exhibits harmonic components at the resonant frequency ω_1 which can occur when ω_1 is located within the control system bandwidth, oscillations are induced in the appendage. In transient operation, these oscillations are not sustained and eventually they die out as a result of the structural damping. But when the control system operation is composed with a succession of transients, like in the presence of environmental disturbances, the modal oscillations may build up to undesirable levels, endangering both control system performance and structural integrity.

A final remark is worthwhile on the stability of a control system in which sensors and actuators are not located on the same rigid section of the structure, as would be the case in Fig. 3 if the feedback loop were closed on θ_1 instead of θ_0. For this case, (6) replaces (4) in the open loop transfer function of the control system. Clearly, a low-modal frequency is definitely destabilizing as it can be seen qualitatively from Fig. 4b which illustrates the effects of the modal factor of (6) on the Nyquist diagram, for the same cases ω_a and ω_b of Fig. 4a. Stability is not affected when the structure is sufficiently rigid, i.e., when the modal frequency is sufficiently higher than the cutoff frequency of the rigid mode transfer function. For sufficiently flexible structures, stability is destroyed by the phase increment of about $-180°$ in a wide range of frequencies, which asymptotically declines to $-90°$, thus making any attempt of compensation very difficult. It is worth noting that the destabilizing effect of the modal frequency considered is linked to the presence of a negative admittance and to the consequent finite pole zero unbalance in the vehicle transfer function θ_1/T_0.

Multiple Control

In designing a control system for a large flexible structure with one or more modal frequencies, one possible configuration is that of a multiple control system with one main loop providing control for the rigid mode, i.e., for the vehicle as a whole, and a number of auxiliary loops providing active damping for the excitable modes.

The following first cut design criterion is suggested. The flexible connections among the bodies are neglected in first approximation and each control loop is designed on the basis of the rigid body dynamics of the component body to which it is directly related, i.e., as if the bodies were independent of each other. Since the degree of this approximation increases with the compliance of the structure, a preliminary design based on the assumption of mechanically uncoupled bodies represents a conservative approach. The bandwidth of each auxiliary loop, however, does not have to be comparable to that of the main control loop. In the course of the analog simulation performed, it has been evident that large sustained oscillations in the structure can be expected when the modal frequencies are well within the main control loop bandwidth. Since the function of each auxiliary loop is only to aid by active means the inadequate passive damping of the structure, a large auxiliary loop bandwidth is not necessary. Thus, while the bandwidth of the main control loop is established by the system's performance specifications, that of each auxiliary loop can effectively be chosen at least one decade smaller than the modal frequency (or group of modal frequencies) to which it is related. The concept is best illustrated with the example given in the next section.

Example—Design and Simulation of a Multiloop System

The structure to be controlled, shown in Fig. 5, has two modal frequencies. The body 0 (inertia J_0) is the main body and its attitude θ_0 is to be controlled with respect to an absolute reference. Two auxiliary control loops are used for controlling the attitude of body 1 and 2 relative to that of body 0.

The following parameters are assumed:

$$J_0 = 1\,sl\text{–}\mathrm{ft}^2;\quad J_1 = 0.5\,sl\text{–}\mathrm{ft}^2;\quad J_2 = 0.25\,sl\text{–}\mathrm{ft}^2$$
$$k_1 = k_2 = k;\quad \alpha_1 = \alpha_2 = \alpha$$

Two typical cases are investigated: 1) $\alpha = 0.1$, $k = 100$, and 2) $\alpha = 0.0001$, $k = 0.0001$. For the given inertias, cases 1 and 2 provide, respectively, modal frequencies of the order of 10 rad/sec and 0.01 rad/sec in each mode, with a damping ratio of approximately $\zeta = 0.005$. Based on its rigid mode alone, the open loop transfer function Y_0 of the main loop (including the sensor and actuator gains and the compensating network) has been assumed

$$Y_0 = 0.1(1+6.66s)/s^2(1+0.666s) \qquad (7)$$

which gives a bandwidth of approximately 0.65 rad/sec and a phase margin of about 54°. Sensor and actuator time constants are neglected here since stability is not the issue. Cases 1 and 2 represent examples in which the highly underdamped modal frequencies occur both within or outside the control system bandwidth. It is expected, therefore, that in the absence of auxiliary control loops, modal oscillations are excited under the action of the main control loop, when the case 2 set of values for α and k are used. For the active damping of these oscillations it is sufficient that each auxiliary loop cutoff

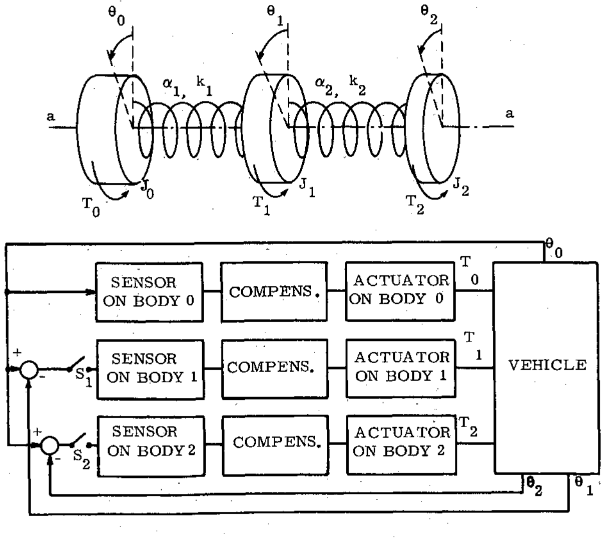

Fig. 5 Multiple control system configuration.

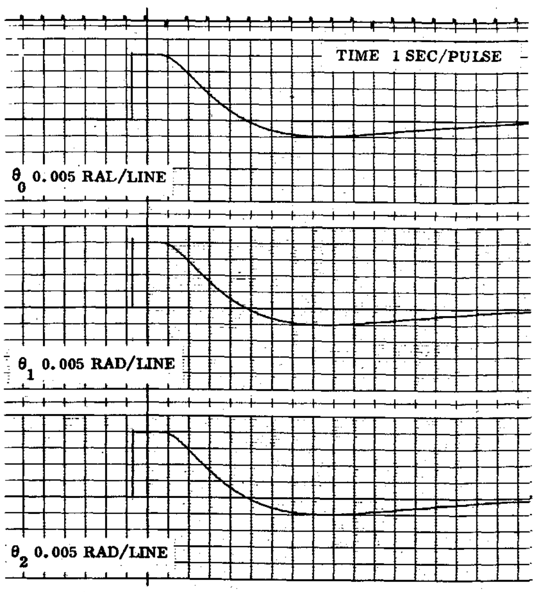

Fig. 6 Single-loop system response for case 1.

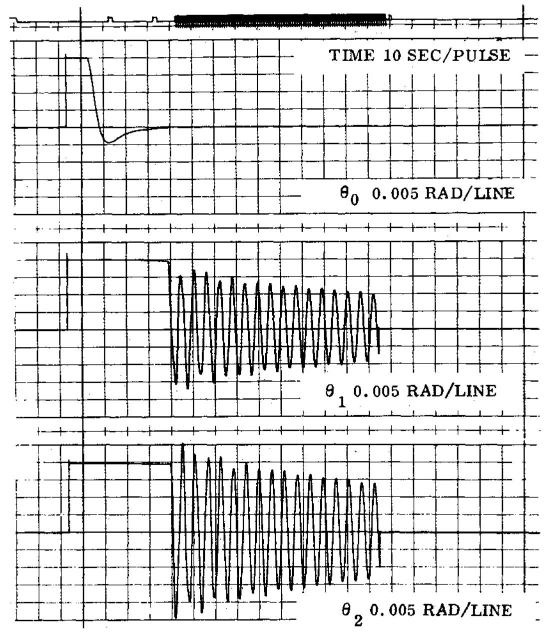

Fig. 7 Single-loop system response for case 2.

frequency be at least one decade smaller than the corresponding model frequency. Thus, with modal frequencies or about 0.01 rad/sec, the assumed auxiliary open loop transfer functions (including compensation) are

$$Y_1 = Y_2 = 0.1M(1+66.6s)/s^2(1+6.66s) \tag{8}$$

where M is the factor by which the auxiliary loop gain is smaller than the main loop gain. For the runs presented in this paper (Fig. 8–10), $M = 0.001$, which yields a bandwidth of 0.003 rad/sec. The compensating network parameters were chosen such as to provide adequate compensation even at higher loop gains, for runs that have not been included in the paper. The runs presented are considered the most significant in that they provide adequate damping of the modal oscillations with a smaller loop gain, which in practice represents a valuable saving in power and weight for the auxiliary control equipment.

The vehicle dynamic equations, used in the simulation, are

$$\begin{aligned} J_0\ddot{\theta}_0+\alpha_1(\dot{\theta}_0-\dot{\theta}_1)+k_1(\theta_0-\theta_1) &= T_0 \\ J_1\ddot{\theta}_1+\alpha_1(\dot{\theta}_1-\dot{\theta}_0)+k_1(\theta_1-\theta_0)+\alpha_2(\dot{\theta}_1-\dot{\theta}_2)+k_2(\theta_1-\theta_2) &= T_1 \\ J_2\ddot{\theta}_2+\alpha_2(\dot{\theta}_2-\dot{\theta}_1)+k_2(\theta_2-\theta_1) &= T_2 \end{aligned} \tag{9}$$

The component transfer functions required to obtain the rigid mode loop transfer functions given by (7) and (8) are listed in Table 1.

The results of the simulation are shown in Figs. 6–10. Figures 6 and 7 show the response of the single-loop control system (switches S1 and S2 of Fig. 5 open) to a step input for the cases 1 and 2, respectively. The excitation of large and very slowly decaying modal oscillations is apparent in Fig. 7. Figures 8 and 9 show the damping effect of auxiliary control on body 1 alone (S1 closed and S2 open) and on body 2 alone (S1 open and S2 closed) respectively, in the system response to a step input for case 2. Figure 10 shows the effect of auxiliary control on both bodies 1 and 2 (S1 and S2 closed) in the step response for case 2. As Figs. 8 and 9 show, the action of only one auxiliary loop on either appendages (1 or 2) is sufficient to produce remarkable improvement in over-all system response. Better results are obtained when both auxiliary loops are active (Fig. 10) as one would expect.

Table 1 Components Transfer Functions

Component	Main loop	Aux. loop
Actuator (lb-ft/V)	1	0.001
Compensation	$0.1\dfrac{1+6.66s}{1+0.66s}$	$0.1\dfrac{1+66.6s}{1+6.66s}$
Sensor (V/Rad)	1	1

Discussion of Results

In discussing attitude control systems for flexible spacecraft, three classes of configurations have been considered: 1) single control systems, with sensors and actuators located at the same point (or on the same rigid component) of the structure; 2) single control systems, with sensors and actuators located at different points (or on different rigid components) of the structure separated by flexible connection; and 3) multiple control systems, with a number of control loops distributed at vital points on the structure.

The analysis and simulation of typical examples built on single-axis, linear, modal representation of a flexible structure have shown the following:

a) When the criterion of mode separability applies, the stability of an attitude control system of the class 1 is, in general, not affected by modal frequencies that are well above or well below the crossover frequency of the rigid mode system approximation. Modal frequencies slightly higher than the cross-

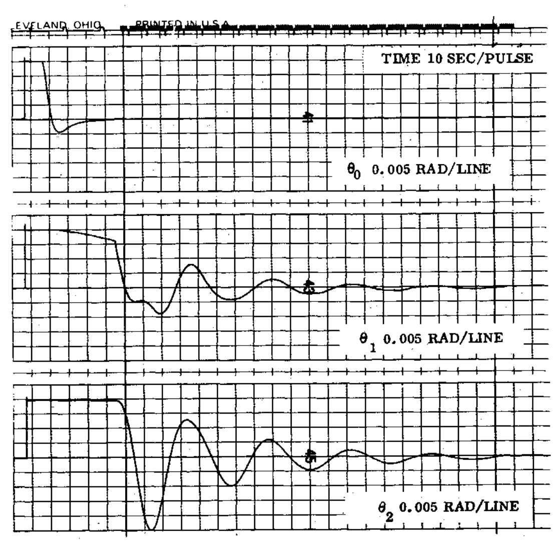

Fig. 8 Case 2: effect of auxiliary control on body 1.

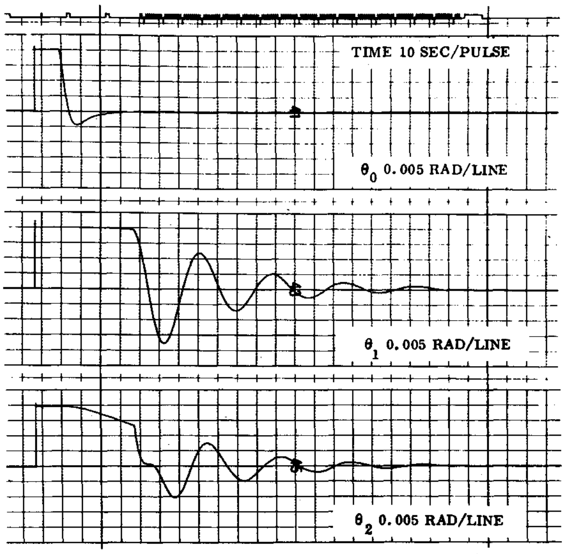

Fig. 9 Case 2: effect of auxiliary control on body 2.

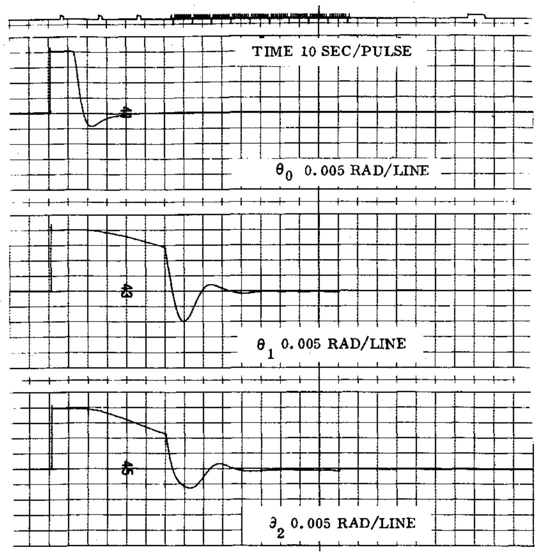

Fig. 10 Case 2: effect of auxiliary control on body 1 and 2.

over frequency may produce stability problems when the time delays within the loop are not negligible. Stability can, however, be preserved by passive compensation.

b) For a class 1 control system, however, large oscillations can be induced in the structure when one or more modal frequencies are located within the system bandwidth. Since the corresponding modal transfer functions are open loop, these oscillations are not due to instability.

c) The presence of negative modal admittances in the structure transfer function and their highly destabilizing effects when the corresponding modal frequencies are lower than the cross-over frequency previously defined are a most serious problem in control systems of the class 2.

d) A control system of the class 3 seems, in principle, to be a good choice for large flexible structures. Such a control system is derived from a class 1 system by the addition of auxiliary control loops as active dampers at critical points on the structure. The performance of a class 3 vs a class 1 system is illustrated by the analog simulation of a typical example in Figs. 6–10.

The practicality of multiple control, however, may be limited to particular types of structures such as those which can be represented with a sufficiently accurate discrete model having a relatively small number of point masses. When such a discretization leads to a large number of point masses, the concept tends to become impractical, as the number of auxiliary loops required may turn out to be exceedingly large.

The practicality of the concept can also be evaluated from another point of view. For structures composed by the flexible interconnection of rigid bodies, it is reasonable to assume that each body can carry the equipment required for the implementation of a control loop, whether primary or auxiliary. If, instead, the structure in question is a large solar array, it is more difficult to think of actuators located at a number of points on the array.

Furthermore, if acceptable control system performance can be obtained by the mere addition of passive dissipative elements at critical points in the structure, the cost and complexity of a multiloop configuration can be avoided. The purpose of this work is only to illustrate the design approach to a multiloop configuration when passive means are inadequate or inapplicable.

The applicability of the criterion of mode separability which is the basis of the stability conclusions drawn above is primarily restricted by the condition that a modal transfer function of the type of Fig. 1 can be obtained. Unfortunately, space vehicles generally cannot be directly represented by accurate or uncoupled modal models of this type. However, warranted by the analytical simplicity and mostly by the possibility of obtaining relatively simple transfer functions which can be effectively used in closed loop control system design, modal models of this type are ordinarily used whenever reasonable approximations are involved. The stability analysis approach discussed in this paper, applicable when, within the limits of validity of a modal model, the separability criterion also applies, is certainly most helpful in preliminary spacecraft control system design which in some cases can be an iterative process. A complete control system simulation, using more accurate structure modeling is eventually required in those cases where significant approximations are involved in the modal representation. An analysis of stability even for the simple case of a valid modal representation whose modes are not separable is more complex, since the modes should be considered all together. The apparent advantage of mode separability is that all structural singularities are located in the left half of the s-plane and are paired in alternate sequence with each pole having an associate zero of slightly lower frequency, as it is shown in the Appendix. This pole-zero pattern makes the analysis approach discussed in the paper possible with the stability conclusions just drawn. If mode separability does not apply, the pole-zero pattern can deviate significantly from the one indicated and the possibility exists that some poles may have associate zeroes at higher frequency. Consequently, the portion of the Nyquist diagram for such a pole-zero pair tends to show destabilizing effects from this mode, when the modal frequency occurs within the control system bandwidth, much in the way indicated in Fig. 4b. Furthermore, if several modes are crowded together the possibility exists that their poles may appear consecutively in the frequency spectrum before their associated zeroes. As one can expect, their impact on stability becomes more critical as their range of frequency approaches the control system cross-over frequency from above. Cases of more complex vehicle representation which deviate from the typical uncoupled modal description of Fig. 1 are analytically more difficult. Extension to these cases of some of the concepts and techniques discussed in this paper is perhaps possible and deserves investigation.

Conclusions

The main objective of this paper is to show the advantages of multiple vs single control in attitude control systems for flexible spacecraft. A secondary objective is to discuss some aspects of the problem of stability and of structure-control system ineraction and of means of analyzing these elements for the particular cases of structures amenable to uncoupled modal representation. It has been shown that when mode separability applies the analysis is simple and clean cut conclusions can be drawn for the control system stability in terms of modal frequency location and control system configuration. The arguments presented are illustrated with two typical examples.

Appendix

The structure transfer function of Fig. 1 can be written as

$$\frac{\theta_q}{T_p} = K_{p,q}{}^0\left[\frac{1}{s^2} + \frac{a_1/\omega_1{}^2}{s^2/\omega_1{}^2+2\zeta_1 s/\omega_1+1}+\ldots+\frac{a_n/\omega_n{}^2}{s^2/\omega_n{}^2+2\zeta_n s/\omega_n+1}\right] \quad \text{(A1)}$$

where $a_i = K_{p,q}{}^i/K_{p,q}{}^0$, $i = 1,2,\ldots,n$. Expression (10) can be reduced to

$$\frac{\theta_q}{T_p} = \frac{A_{2n}s^{2n}+A_{2n-1}s^{2n-1}+\ldots+A_2 s^2+A_1 s+A_0}{s^2(s^2/\omega_1{}^2+2\zeta_1 s/\omega_1+1)\ldots(s^2/\omega_n{}^2+2\zeta_n s/\omega_n+1)} \quad \text{(A2)}$$

where

$$A_{2n} = (1+a_1+a_2+\ldots+a_n)/\omega_1{}^2\omega_2{}^2\ldots\omega_n{}^2 \quad \text{(A3a)}$$

$$A_{2n-1} = \frac{2\zeta_1}{\omega_1}\left(\frac{1+a_2+a_3+\ldots+a_n}{\omega_2{}^2\omega_3{}^2\ldots\omega_n{}^2}\right)+\frac{2\zeta_2}{\omega_2}\left(\frac{1+a_1+a_3+\ldots+a_n}{\omega_1{}^2\omega_3{}^2\ldots\omega_n{}^2}\right)+\ldots+\frac{2\zeta_n}{\omega_n}\left(\frac{1+a_1+a_2+\ldots+a_{n-1}}{\omega_1{}^2\omega_2{}^2\ldots\omega_{n-1}{}^2}\right) \quad \text{(A3b)}$$

$$A_{2n-2} = \frac{4\zeta_1\zeta_2}{\omega_1\omega_2}\left(\frac{1+a_3+a_4+\ldots+a_n}{\omega_3{}^2\omega_4{}^2\ldots\omega_n{}^2}\right)+\frac{4\zeta_1\zeta_3}{\omega_1\omega_3}\left(\frac{1+a_2+a_4+\ldots+a_n}{\omega_2{}^2\omega_4{}^2\ldots\omega_n{}^2}\right)+\ldots+\frac{4\zeta_{n-1}\zeta_n}{\omega_{n-1}\omega_n}\left(\frac{1+a_1+a_2+\ldots+a_{n-2}}{\omega_1{}^2\omega_2{}^2\ldots\omega_{n-2}{}^2}\right)+\frac{1+a_2+a_3+\ldots+a_n}{\omega_2{}^2\omega_3{}^2\ldots\omega_n{}^2}+\frac{1+a_1+a_3+\ldots+a_n}{\omega_1{}^2\omega_3{}^2\ldots\omega_n{}^2}+\ldots+\frac{1+a_1+a_2+\ldots+a_{n-1}}{\omega_1{}^2\omega_2{}^2\ldots\omega_{n-1}{}^2} \quad \text{(A3c)}$$

$$A_2 = \frac{4\zeta_1\zeta_2}{\omega_1\omega_2}+\frac{4\zeta_1\zeta_3}{\omega_1\omega_3}+\ldots+\frac{4\zeta_{n-1}\zeta_n}{\omega_{n-1}\omega_n}+\frac{1+a_1}{\omega_1{}^2}+\ldots+\frac{1+a_n}{\omega_n{}^2} \quad \text{(A3d)}$$

$$A_1 = 2\zeta_1/\omega_1+2\zeta_2/\omega_2+\ldots+2\zeta_n/\omega_n \quad \text{(A3e)}$$

$$A_0 = 1 \quad \text{(A3f)}$$

When the admittances $K_{p,q}{}^i$ are all positive, the A's are all positive and the numerator polynomial of (A2) is complete and or order $2n$. For purpose of analysis, it is convenient to have the numerator of (A2) in factorized form. Assume the form

$$\left(\frac{s^2}{\omega_1{}'^2}+\frac{2\zeta_1{}'s}{\omega_1{}'}+1\right)\ldots\left(\frac{s^2}{\omega_n{}'^2}+\frac{2\zeta_n{}'s}{\omega_n{}'}+1\right) = B_{2n}s^{2n}+B_{2n-1}s^{2n-1}+\ldots+B_2 s^2+B_1 s+B_0 \quad \text{(A4)}$$

where

$$B_{2n} = 1/\omega_1{}'^2\omega_2{}'^2\ldots\omega_n{}'^2 \quad \text{(A5a)}$$

$$B_{2n-1} = \frac{2\zeta_1{}'}{\omega_1{}'}\left(\frac{1}{\omega_2{}'^2\omega_3{}'^2\ldots\omega_n{}'^2}\right)+\frac{2\zeta_2{}'}{\omega_2{}'}\left(\frac{1}{\omega_1{}'^2\omega_3{}'^2\ldots\omega_n{}'^2}\right)+\ldots+\frac{2\zeta_n{}'}{\omega_n{}'}\left(\frac{1}{\omega_1{}'^2\omega_2{}'^2\ldots\omega_{n-1}{}'^2}\right) \quad \text{(A5b)}$$

$$B_{2n-2} = \frac{4\zeta_1{}'\zeta_2{}'}{\omega_1{}'\omega_2{}'}\left(\frac{1}{\omega_3{}'^2\omega_4{}'^2\ldots\omega_n{}'^2}\right)+\frac{4\zeta_1{}'\zeta_3{}'}{\omega_1{}'\omega_3{}'}\left(\frac{1}{\omega_2{}'^2\omega_4{}'^2\ldots\omega_n{}'^2}\right)+\ldots+\frac{4\zeta_{n-1}{}'\zeta_n{}'}{\omega_{n-1}{}'\omega_n{}'}\left(\frac{1}{\omega_1{}'^2\omega_2{}'^2\ldots\omega_{n-2}{}'^2}\right)+\frac{1}{\omega_2{}'^2\omega_3{}'^2\ldots\omega_n{}'^2}+\frac{1}{\omega_1{}'^2\omega_3{}'^2\ldots\omega_n{}'^2}+\ldots+\frac{1}{\omega_1{}'^2\omega_2{}'^2\ldots\omega_{n-1}{}'^2} \quad \text{(A5c)}$$

$$B_2 = \frac{4\zeta_1{}'\zeta_2{}'}{\omega_1{}'\omega_2{}'}+\frac{4\zeta_1{}'\zeta_3{}'}{\omega_1{}'\omega_3{}'}+\ldots+\frac{4\zeta_{n-1}{}'\zeta_n{}'}{\omega_{n-1}{}'\omega_n{}'}+\frac{1}{\omega_1{}'^2}+\ldots+\frac{1}{\omega_n{}'^2} \quad \text{(A5d)}$$

$$B_1 = 2\zeta_1{}'/\omega_1{}'+2\zeta_2{}'/\omega_2{}'+\ldots+2\zeta_n{}'/\omega_n{}' \quad \text{(A5e)}$$

$$B_0 = 1 \quad \text{(A5f)}$$

For (A4) to represent the numerator of (A2), each B_j ($j = 1,2,\ldots,2n$) must equal the corresponding A_j. It can be seen readily that under the condition

$$0 < a_i \ll 1 \qquad i = 1,2,\ldots,n \quad \text{(A6)}$$

the system of $2n$ equations $B_j = A_j$ in the $2n$ unknowns ω_i', ζ_i' has the approximate solution

$$\omega_i'^2 = \omega_i^2/(1+a_i); \quad \zeta_i'/\omega_i' = \zeta_i/\omega_i \qquad i = 1,2,\ldots,n \quad \text{(A7)}$$

This is based on the approximation

$$\prod_k\left(\frac{1}{\omega_k'^2}\right) = \prod_k\left(\frac{1+a_k}{\omega_k^2}\right) \simeq \left[1+\sum_k(a_k)\right]\Big/\prod_k(\omega_k^2)$$

valid under condition (A6).

The condition under which (A7) are sufficiently approximated can be defined as mode separability. When this applies, each structural pole has an associates zero which occurs at slightly lower frequency, as it is shown by (A7), and is also located in the left half of the s-plane. When sufficient mode separation occurs, the effect of each mode on control system stability can be evaluated separately. The effect of each mode on the Nyquist diagram takes the form illustrated in Fig. 4a. From a practical point of view, mode separability also implies that no two modal frequencies be nearly the same.

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